Eigenvalues and Eigenvectors

Def: Let A be an $n \times n$ matrix. Suppose that $\vec{x}$ is in $\mathbb{R}^n$ and $\vec{x} \neq \vec{0}$ and λ is in $\mathbb{R}$. If $A\vec{x} = \lambda\vec{x}$ then we call λ an eigenvalue of A, and $\vec{x}$ an eigenvector of A corresponding to λ, Given an eigenvalue λ of A, the set of all $\vec{x}$ with $A\vec{x} = \lambda\vec{x}$ is called the eigenspace of A corresponding to λ, and is denoted by $E_\lambda(A)$. That is $E_\lambda(A) = \{x \mid \vec{x} \in \mathbb{R}^n \text{ and } A\vec{x} = \lambda\vec{x}\}$

note: $\vec{0}$ is in $E_\lambda(A)$ to make it a subspace even though $\vec{0}$ isn't an eigenvector

Ex: $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\lambda = 3$

So, $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in E_3(A)$, since $A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

How do we find the eigenvalues and eigenvectors of A?

Suppose $\vec{x} \neq \vec{0}$ and $A\vec{x} = \lambda\vec{x}$, Then $(A - \lambda I)\vec{x} = \vec{0}$. Since $\vec{x} \neq \vec{0}$ we have $(A - \lambda I)\vec{x} = \vec{0}$. Hence a nonzero solution $(A-\lambda I)^{-1}$ does not exist. So $\det(A - \lambda I) = 0$.

So to find the eigenvalues of A we solve the equation $\boxed{\det(A - \lambda I) = 0}$. This is called the characteristic equation (or polynomial) of A.

Ex: Find the eigenvalues of $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8-\lambda \end{pmatrix} = -\lambda^3 + 8\lambda^2 - 17\lambda + 4$$

$$= -(\lambda - 4)\underbrace{(\lambda^2 - 4\lambda + 1)}_{\text{Has solutions } \lambda = 2 \pm \sqrt{3}}$$

So, the eigenvalues are $\lambda = 4, 2+\sqrt{3}, 2-\sqrt{3}$.

How do we find the eigenvectors that correspond to say $\lambda = 4$?

We solve the equation $A\vec{x} = 4\vec{x}$,

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 4\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longrightarrow \begin{pmatrix} x_2 \\ x_3 \\ 4x_1 - 17x_2 + 8x_3 \end{pmatrix} = \begin{pmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{pmatrix}$$

$$\longrightarrow \begin{aligned} -4x_1 + x_2 \qquad &= 0 \\ -4x_2 + x_3 &= 0 \\ 4x_1 - 17x_2 + 4x_3 &= 0 \end{aligned} \longrightarrow \left(\begin{array}{ccc|c} -4 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 4 & -17 & 4 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 + R_3 \to R_3} \left(\begin{array}{ccc|c} -4 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & -16 & 4 & 0 \end{array}\right) \xrightarrow{-4R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} -4 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1/4 & 0 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left.\begin{aligned} x_1 - \tfrac{1}{4}x_2 &= 0 \\ x_2 - 1/4\, x_3 &= 0 \end{aligned}\right\} \quad \begin{aligned} x_1 &= 1/16\, t \\ x_2 &= 1/4\, t \\ x_3 &= t \end{aligned}$$

~~So the eigenspace corresponding to $\lambda = 4$ contains the eigenvectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} 1/16 \\ 1/4 \\ 1 \end{pmatrix}$. For ex, $\begin{pmatrix} 1/16 \\ 1/4 \\ 1 \end{pmatrix}$ is in there.~~

See next

Therefore, the eigenspace corresponding to $\lambda = 4$ consists of all the vectors of the form $\begin{pmatrix} t/16 \\ t/4 \\ t \end{pmatrix} = t\begin{pmatrix} 1/16 \\ 1/4 \\ 1 \end{pmatrix}$.

That is,

$$E_4(A) = \left\{ t\begin{pmatrix} 1/16 \\ 1/4 \\ 1 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

$$\boxed{\text{Theorem: Let } A \text{ be an } n\times n \text{ matrix. Let } \lambda \text{ be an eigenvalue of } A. \text{ Then } E_\lambda(A) \text{ is a subspace of } \mathbb{R}^n.}$$

In the previous example note that a basis for $E_4(A)$ is $\left\{ \begin{pmatrix} 1/16 \\ 1/4 \\ 1 \end{pmatrix} \right\}$.

Hence $E_4(A)$ is 1-dimensional.

Def: Let A be an $n \times n$ matrix and λ an eigenvalue of A.
The algebraic multiplicity of λ is the multiplicity of λ as a root of the characteristic polynomial of A.
The geometric multiplicity of λ is the dimension of the eigenspace $E_\lambda(A)$.

Ex: $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$ $\lambda = 4$

The characteristic poly. of A is
$\det(A - \lambda I) = -(\lambda - 4)(\lambda - (2+\sqrt{3}))(\lambda - (2-\sqrt{3}))$

So, $\lambda = 4$ has algebraic multiplicity 1.
Since $E_4(A)$ has dimension 1, the geometric multiplicity is also 1.

Ex: Find bases for the eigenspaces of

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

Find the geometric and algebraic multiplicities of each eigenvalue.

$$\det(A - \lambda I) = -(\lambda^3 - 5\lambda^2 + 8\lambda - 4) = 0$$
$$= -(\lambda - 1)(\lambda - 2)^2$$

$\lambda = 1, 2$ are the eigenvalues.

$\lambda = 2$: A basis is $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = 1$: A basis is $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

λ	alg. mult	geo. mult.
2	2	2
1	1	1

~~Thm: Suppose that λ is an eigenvalue of A and $\vec{x} \neq \vec{0}$ is an eigenvector. Then~~

Theorem: Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. Then,

$$\begin{pmatrix} \text{geometric mult.} \\ \text{of } \lambda \end{pmatrix} \leq \begin{pmatrix} \text{alg. mult.} \\ \text{of } \lambda \end{pmatrix}.$$

Ex: $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{pmatrix}$

$\det(A - \lambda I) = -(\lambda - 1)(\lambda - 2)^2$

$\lambda = 1$ basis is $\begin{pmatrix} 1/8 \\ -1/8 \\ 1 \end{pmatrix}$

$\lambda = 2$ basis is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

λ	alg. mult	geo. mult
1	1	1
2	2	1

Diagonalizing a matrix

<u>Def:</u> An $n \times n$ matrix A is called <u>diagonalizable</u> if there is an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. The matrix P is said to <u>diagonalize</u> A.

<u>Ex:</u> $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

$\lambda = 2 \rightarrow$ basis for eigenspace $p_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $p_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = 1 \longrightarrow$ basis for eigenspace $p_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

Let $P = \left(p_1 \,\middle|\, p_2 \,\middle|\, p_3 \right) = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$P^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix}$. Can calculate that

$$P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

What is happening here?

(Anton pg 365)

Theorem Let A be an $n \times n$ matrix. Then the following are equivalent:

① A is diagonalizable

② A has n linearly independent eigenvectors.

③ All the eigenvalues of A satisfy (geometric multiplicity of λ) = (alg. mult. of λ).

proof:

② $\Rightarrow$ ① Suppose A has n linearly independent eigenvectors $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (the eigenvalues can repeat here).

Let $P = \left(\vec{p}_1 \middle| \vec{p}_2 \middle| \ldots \middle| \vec{p}_n \right)$. Then

$$AP = \left(A\vec{p}_1 \middle| A\vec{p}_2 \middle| \ldots \middle| A\vec{p}_n \right) = \left(\lambda_1 \vec{p}_1 \middle| \lambda_2 \vec{p}_2 \middle| \ldots \middle| \lambda_n \vec{p}_n \right)$$

$$= \left(\vec{p}_1 \middle| \vec{p}_2 \middle| \ldots \middle| \vec{p}_n \right) \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

$$= PD,$$

where D is the diagonal matrix with $\lambda_1, \lambda_2, \ldots, \lambda_n$ on the diagonal. Since the columns of P are linearly independent, $\operatorname{rank}(P) = n$ and so P^{-1} exists. So, $P^{-1}AP = D$.

Use $\operatorname{rank}(P) + \operatorname{nullity}(P) = n$
So $\operatorname{nullity}(P) = 0$
So, P^{-1} exists

Procedure to diagonalize an $n \times n$ matrix with n linearly independent eigenvectors

① Find n linearly independent eigenvectors of A, say $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n$. (if you can)

② Let P be the matrix having $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n$ as its column vectors.

③ The matrix $P^{-1}AP$ will be diagonal with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as the diagonal entries where λ_i is the eigenvalue of A corresponding to $\vec{p}_i$.

Ex: $A = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}$

$P = \begin{pmatrix} 1/3 & 0 \\ 1 & 1 \end{pmatrix}$ $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\left.\begin{array}{l}
\textbf{Ex:}\ A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{pmatrix} \quad \det(A - \lambda I) = -(\lambda-1)(\lambda-2)^2 \\
\lambda = 1, 2 \quad \underline{\lambda = 1}\ \text{one-dim eigenspace with basis } \vec{p}_1 = \begin{pmatrix} 1/8 \\ -1/8 \\ 1 \end{pmatrix} \\
\qquad\qquad \underline{\lambda = 2}\ \text{one-dim eigenspace with basis } \vec{p}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
\text{Only 2 linearly independent eigenvectors. So cannot diagonalize } A.
\end{array}\right\}\ \text{example from before}$$